How do Students Create Algorithms? Exploring a Group's Attempt to Maximise Happiness

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This paper addresses the need for empirical research on the processes by which students create algorithms. I analyse the collaborative work of three high-school students on a contextualised graph theory task, in which they created an algorithm for maximising the happiness score of a seating arrangement. The group found an optimal arrangement but created an algorithm that did not fully account for this arrangement. The group's written algorithm reflected only the properties of their optimal arrangement that they explicitly noticed after creating the arrangement. And, these explicitly-noticed properties aligned with the group's predominant contextual considerations.

Discrete mathematics – "the math of our time" Dossey (1991, p. 1) – is ever-growing in prominence due to its significance in computer science and the many real-world applications of its sub-branches (e.g., probability, logic, combinatorics, and cryptography). Accordingly, mathematics education research has sought to promote the teaching and learning of discrete mathematics (Hart & Sandefur, 2017; Kenney & Hirsch, 1991).

At the heart of discrete mathematics lies *the algorithmic approach* (Kenney & Hirsch, 1991) which entails solving a problem by devising and analysing an algorithm that constructs a solution. The algorithmic approach distinguishes discrete mathematics from traditional mathematics such as algebra, calculus, and topology (Kenney & Hirsch, 1991). Thus, developing students' competence with the algorithmic approach is a central theme in discrete mathematics education research (Morrow & Kenney, 1998; Hart & Sandefur, 2017).

Past research aimed at promoting students' competence in the algorithmic approach has largely focused on: 1) explicating the processes by which experts create algorithms (e.g., Weintrop et al., 2016); and 2) designing tasks (henceforth referred to as *algorithmic tasks*) that require students to create their own algorithms (e.g., see Morrow & Kenney, 1998 for numerous examples). The overarching goal of these two lines of research is more or less to identify *how the experts operate*, and then create tasks in which students are expected to operate in similar ways. While this past research has been useful in advancing our understanding of how experts create algorithms, it has lacked attention to the processes by which students create their algorithms.

The study reported in this paper is a first attempt to address the foregoing gap in the literature, by exploring the process through which a group of three students create an algorithm in a contextualised graph theory task. The aims of this study are to: 1) explore the idiosyncratic interpretations that students employ for the various requests of the task (e.g., the request to find a solution for the case at hand; the request to create an algorithm that finds a solution); 2) explore how these interpretations interact; and 3) discuss how the eventual algorithm reflects (or does not reflect) these interpretations.

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Background Literature

The need to provide students with more opportunities to engage with non-routine algorithmic tasks, and to equip them with the tools to create their own algorithms has been well-acknowledged in the literature (Hart & Sandefur, 2017). Two predominant lines of research have emerged in response to this need. The first line of research (e.g., Weintrop et al., 2016) has revealed common characteristics (e.g., recursion and induction) underlying the processes by which experts create algorithms. These processes are then construed as the processes that students *should* engage with when they work on algorithmic tasks. Furthermore, characteristics of the experts' algorithms, such as efficiency and generalisability, are taken to be the measures of quality for assessing students' algorithms. While this line of research is useful in providing ideals to which students could aspire, it has been acknowledged that it is unreasonable to evaluate students' final algorithms with respect to expert-qualities, without understanding the processes by which students' algorithms emerged (Hart & Sandefur, 2017). The second line of research (see Morrow & Kenney, 1998 for numerous examples) complements the first by designing algorithmic tasks and discussing the potential processes that students could engage in when working on these tasks. Despite the usefulness of this line of research, the hypothetical student activity that guides the design of these environments is potentially limiting, due to disparities between what students are expected to do, and what students actually do on these tasks (e.g., see Cai et al., 1998).

In a rare empirical study that analysed how students create algorithms, Cai et al. (1998) distinguished between *a procedure* and *an algorithm*. The former refers to the entire process by which a solution is found, while the latter refers to communicating the process in a succinct way which utilises recurring steps and patterns in the process. For instance, consider the problem: *how many days are there between March 24th and April 21st of any given year*? A procedure involves, say, counting every day between the two dates, and coming up with 28 days. An algorithm involves, say, noticing the recurrence of seven days between these two dates, and thus getting to 28 days by way of 7 x 4. Cai et al. observed that the students easily constructed a procedure but struggled with creating an algorithm. The students regurgitated their procedure when asked for an algorithm, and there was no evidence that the students noticed any recurring patterns in their procedure. Studies such as Cai et al. (1998) reveal nuances of the algorithmic approach (or the process of creating algorithms) that perhaps experts take for granted, but with which students might struggle. The study reported in this paper can be situated within research such as Cai et al. (1998) that seeks to expose, by way of empirical data on student activity, nuances of the algorithmic approach, and propose ways for helping students cope with these nuances.

Theoretical Framework: Considerations of Aptness

In the context of tasks asking participants to pose mathematical problems, Kontorovich (2016) discussed the construct of *considerations of aptness* "to capture uncertainties and doubts of a poser together with the meaning that is eventually attributed to the vague terms [stated in the task instructions]" (p. 246). The construct comprises five types of considerations (see Kontorovich, 2016), but for my study I focus on one of them: *considerations of aptness to the task*, which "is concerned with the poser's attempt to satisfy explicit requests of the given stimulus" (p. 246). The use of considerations of aptness in this study is motivated by the aim to explore the idiosyncratic interpretations that students employ for the various requests of the task.

I use *considerations of aptness to the task* to refer broadly to the students' attempts to fulfil the multiple requests of the task. The students' interpretations of these requests can be inferred from their idiosyncratic attempts to satisfy the explicit requests. One thing to note, however, is that the algorithmic task (used in this study) contains multiple explicit requests such as: finding an optimal solution for a given case; creating a method for finding an optimal solution for the general case; and framing the method as a letter. Thus, there will be different types of considerations of aptness to the task, depending on which particular explicit request the students are addressing.

Research Questions

The overall aim of the study (to reiterate) is to explore the process by which a group of students create an algorithm in a contextualised graph theory task. More specifically, through the lens of considerations of aptness to the task, the study explores the following questions: 1) What considerations of aptness to the task does the group employ? 2) How do these considerations of aptness interact? 3) How are these considerations of aptness reflected (or not reflected) in their final algorithm?

Method

The data for this study is taken from the collaborative work of three students who at the time of data collection, were in a Year 12 mathematics (calculus) class at a high school in New Zealand. All three students knew each other well and were recruited as part of a larger research project that explores students' engagement with discrete mathematics through contextualised tasks (Yoon, Chin, Griffith Moala, & Choy, 2017). The group worked on *The Birthday Seating Task* (Davies, Chin, Griffith Moala, & Yoon, 2016; adapted from https://xkcd.com/173), incorporating the theme of optimisation which is one of the central themes in all areas and contexts of discrete mathematics (see Hart & Sandefur, 2017). The fifty-minute session took place outside of class time and was video-recorded. The group worked in the presence of an interviewer who answered clarification questions but avoided providing mathematical hints.

The task begins with some warm up questions that familiarise students with weighted graphs (networks) in the context of different relationships. For example, in the graph below (Figure 1), the nodes represent people, and the number on an edge between two nodes represents a "happiness score" for the corresponding people's relationship. After completing the warm-up questions, the students are given the following scenario:

> Michael is turning 15 and has decided to invite his friends to the movies this weekend. He creates a Facebook event and invites his best friends. 7 friends have confirmed that they will attend. Michael decides to make a seating plan beforehand as the cinemas will only provide them with one row of seats, and he knows some of his friends don't get along. He draws a graph that represents the relationships between the seven confirmed friends [Figure 1] and shows the happiness scores between each pair of people [NB: if there's no edge between a pair of nodes (e.g., A & E or D & F) then you can assume that their happiness score is 0] (Davies et al., 2016, p. 9).

The instructions for the task are (stated below). After reading the instructions, the group the group were told to work together and that they must agree on everything that they include in their algorithm.

> Create an algorithm (method) that Michael can use to find the best seating arrangement (i.e., the one with the highest total happiness score) for his friendship graph. Remember all of Michael's friends must sit in one row at the cinema. Write a letter to Michael in which you:1) State the best seating arrangement; 2) Explain your method for choosing the best seating arrangement, and how/why it is

guaranteed to give you the best seating arrangement; 3) Describe how Michael can adapt your method to choose a seating arrangement if more of his friends (other than the 7 given in the graph) show up unexpectedly at the cinema. Remember that some of the unexpected friends might not get along with. some of the 7 confirmed friends (Davies et al., 2016, p. 10).

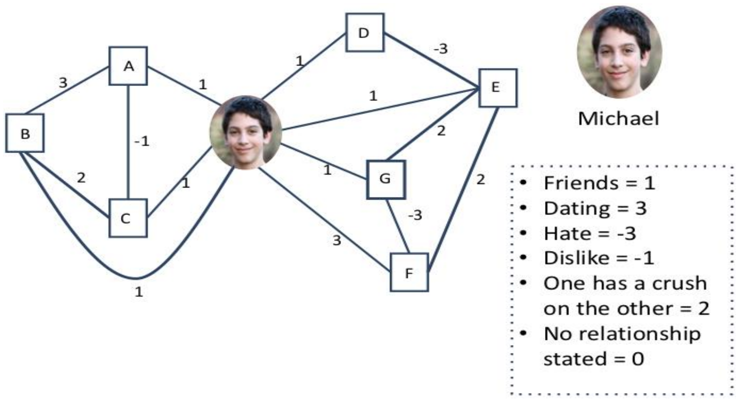

Figure 1. Michael's friendship graph.

To analyse the data, I first read the annotated transcript of the group's work and identified which requests of the task (e.g., the request to create an algorithm; the request to find the best seating arrangement) they were addressing and divided the transcript into excerpts corresponding to these different requests. I then compared these excerpts and sorted them into four basic categories that I interpreted as the primary considerations of aptness to the task that the group employed. These were considerations of aptness to: (i) the best seating arrangement for the given graph; (ii) a method for finding the best seating arrangement for the given graph; (iii) the real-world context of the task (i.e., a group of people with particular relationships going to the movies); and (iv) the unexpected friends. I then explored the interactions among these four considerations of aptness throughout the group's work and examined how the considerations were reflected (or not reflected) in the final algorithm. In the next section, I give an *account-of* (Mason, 2002) the group's work, summarising *what happened* in the entire session. Then, I describe two themes that emerged from the analysis. Throughout the session, the students and interviewer switched between "algorithm" and "method", and I preserve both when describing and analysing their work.

A Summary of the Group's Work

The group begins by agreeing that, "Michael should be in the middle, because it's his party." They then discuss how "the two people that hate each other the most" should be seated at the ends of the row, "that way they are farthest away from each other" and they don't "spoil things". Then, Sia notices that Michael is the only person with whom *D* has a "positive relationship." They remark that perhaps *D* should sit next to Michael, because that is "where *D* would feel most comfortable [and] if you don't put *D* next to Michael, he's just not going to have any fun, because he hates *E*, and no one else knows him." Then, each student creates a seating arrangement. Sia and Para both create (separately) *G-E-F-M-D-A-B-C* which has a score of 13, and Heti creates: *A-B-C-M-F-E-G-D* which also has a score of 13 (note, the decisions they make while they are creating these arrangements are not evident in the data). They examine the arrangements and say, "they both don't have any negatives”. Para then asks, “how about the ones who show up unexpectedly?” Sia responds, “we’ll just assume that they get along with the people at the ends of the seating row”. Para adds, “yeah, it’s kind of rude to show up unexpectedly right?” Heti says, “Yeah, why couldn’t they just confirm that they will come? It’s so inconvenient!” Para continues, “OK, so just put them on the edges.” Then, Para says, “It would be so much better if we just gave all of them tickets, and then they choose where they want to sit.” Heti nods and Sia responds, “Nah, because if you think about it that’ll be awkward if everyone chooses, because it’s not just a party, it’s the movies. So, what if like *E* comes and sits next to *F*, and then *B* also comes and sits next to *F*? Then Michael can’t sit next to *F*, but *F* and Michael is a three, and *B* and *M* is only 1.” Para argues, “But if they like each other then they will sit together.” Sia responds, “Yeah but that won’t always happen if everyone gets to choose where they sit.”

Para reads the task instructions aloud, and says, “OK so we need to explain our method.” Heti starts writing, then asks, “which one [of their two arrangements above] should we give?” Sia says, “this one (*G-E-F-M-D-A-B-C*) right? Because we want *D* next to Michael. ‘cause that’s where *D* will feel most comfortable.” They further endorse their seating arrangement by saying that it “has no negative relationships”, and every person in the arrangement sits next to at least one person with whom s/he has a positive relationship, “so everyone has a good time”. Heti then asks, “so our method was…?”. Para responds, “our two goals was [*sic*], keep the negatives ones away from each other, and keep *D* next to Michael.” Both Sia and Heti nod in agreement. Heti finishes writing the letter [see Figure 2]. Sia then looks at the task instructions again, and says “Oh man it’s that thing again, we have to like think about a different situation, and show that our method still works”. Heti and Para both say, “We’ve already done that. We’ve said just assume they get along with *G* and *C*.” Sia says: “Oh yes, that’s right. Cool.”

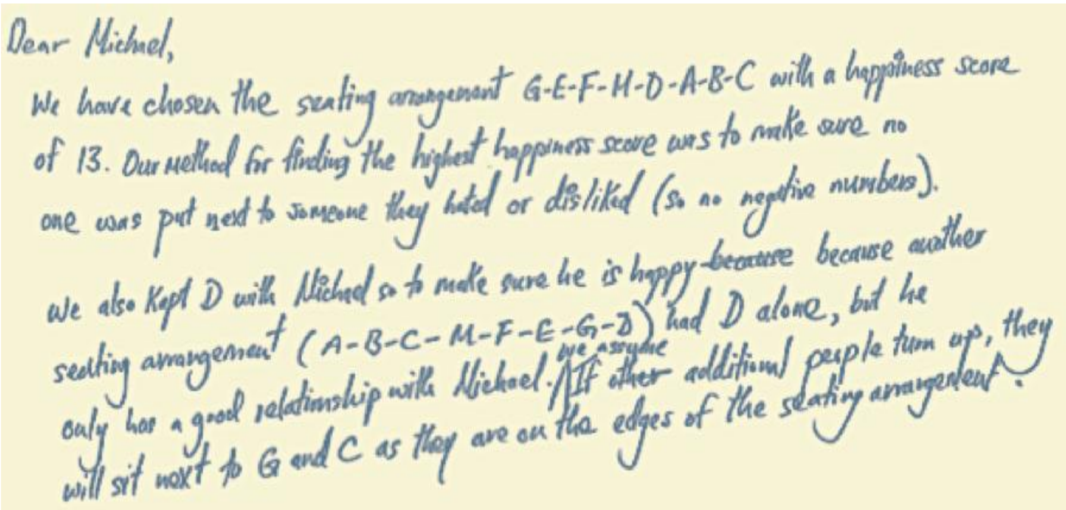
Dear Michel,
We have chosen the seating arrangement G-E-F-M-D-A-B-C with a happiness score
of 13. Our method for finding the highest happiness score was to make sure no
one was put next to someone they hated or disliked (so no negative numbers).
we also kept D with Michael so to make sure he is happy ~~because~~ because another
seating arrangement (A-B-C-M-F-E-G-D) had D alone, but he
only has a good relationship with Michael. we assume If other additional people turn up, they
will sit next to G and C as they are on the edges of the seating arrangement.

Figure 2. The group’s letter containing their algorithm.

Findings

Two main themes emerged from my analysis of the students’ considerations of aptness. I describe these two themes in turn.

The predominance of contextual considerations

From the outset, the group's considerations of aptness to the real-world context is evident. For instance, they say that Michael should sit in "the middle because it is his party." And, "the two people who hate each other the most [should sit] on the ends of the row" so that they don't "spoil things". Furthermore, after noticing that person *D* had a positive relationship with only Michael, they decided to seat *D* next to Michael because "that is where *D* would feel most comfortable" and "otherwise he just won't have any fun." These considerations to the real-world context seem to influence the group's considerations of aptness to the best seating arrangement. Though the two seating arrangements they created (*A-B-C-M-F-E-G-D* and *G-E-F-M-D-A-B-C*) were equivalent in terms of *total happiness score*, the group endorsed the latter because: *D* sits next to Michael ("where he [*D*] feels most comfortable"); and every person in the arrangement sits next to a person with whom s/he has a positive relationship ("so everyone has a good time"). Competition between considerations of aptness to the real-world context and considerations of aptness to a method for finding the maximal seating arrangement is noticeable when the group address the issue of the unexpected friends. Para remarked that "it would so much easier if we just give them tickets and they choose where to sit", but Sia countered with a scenario exemplifying how Para's suggestion might not yield the highest happiness score. Ultimately, the group's considerations of aptness to the real-world context impact their considerations to the unexpected friends, as evidenced by their remarks: "seat them on the ends of the row" because "it's kind of rude to show up expectedly" and "it's so inconvenient".

The algorithm reflects only explicitly-noticed properties of the optimal arrangement

The group's letter contains a seating arrangement, *G-E-F-M-D-A-B-C*, which they endorsed as the best, and a method for finding the best seating arrangement. I infer strictly from their letter that the group's method comprises three rules: 1) avoiding negative relationships; 2) keeping D next to Michael; 3) placing the unexpected friends on the edges of the optimal arrangement. These three rules align with the group's predominant contextual considerations. For instance, "avoiding negatives" aligned with *everyone having a good time*, and "keeping D next to Michael" aligned with *making D feel comfortable*. These three rules (particularly the first two), however, do not fully account for the group's optimal arrangement: That is, creating an arrangement for the given graph using only these two rules, would not necessarily yield the group's optimal arrangement. For example, the two arrangements *C-B-A-M-D-G-E-F* and *B-E-G-C-F-M-D-A* (among others), both of which have lesser happiness scores (11 and 6 respectively) can be obtained via these two rules. Evidently, the group's optimal arrangement has particular properties that distinguish it from these two arrangements. One such distinguishing property is: *each person sits next to a person with whom s/he has the highest relationship* (*or the next highest if the highest one is taken*). The group's final algorithm does not account for such distinctions.

What may have led to the emergence of the group's written algorithm (i.e., in particular, one which does not fully account for their optimal arrangement)? To answer this question, I note that the group's algorithm was written after they created the optimal seating arrangement. Further, the three rules in group's algorithm can be traced to explicit remarks made after the creation of the optimal seating arrangement. For example, *avoiding negatives* can be traced to the remark that the optimal arrangement had no negative edges. Also, *keeping D next to Michael* and placing the *unexpected the friends on the edges* can both be traced to the group's aforementioned considerations of aptness to the real-world context. In contrast, the group did not make any explicit remarks regarding, for instance, how each person in the optimal seating arrangement sits next to a person with whom s/he has the highest relationship (or the next best if the highest person has already been taken). It thus seems that the final algorithm reflects only those properties of the optimal solution that the group explicitly noticed after the creating the arrangement. And, as mentioned above, these explicitly-noticed properties were ones that aligned with their predominant contextual considerations. This might suggest that the group considered the aptness of the optimal arrangement primarily with respect to the real-world context.

Discussion

The two main findings of this paper were: 1) the group's final algorithm consisted of three rules that reflected only the properties of the optimal arrangement that the group explicitly noticed (after creating it); and 2) the explicitly-noticed properties aligned with the group's predominant considerations of aptness to the real-world context. These two findings help explain how the group's algorithm (in particular, the two rules of "avoiding negatives" and "keeping *D* with Michael") did not fully account for their optimal arrangement. For instance, one distinguishing property of the optimal arrangement that the group's algorithm did not account for was: *each person sits next to a person with whom s/he has the highest relationship (or the next highest if the highest one is taken*). I infer that although the actual rules that guided the group's decisions while they were creating the arrangement were not explicit in the data, discrepancies exist between the actual rules used and the rules communicated in the algorithm. From this inference, I claim that the unaccounted-for distinguishing property emerged via a rule that was not expressed in the final algorithm. Such a rule could be that of "maximising locally", which stipulates that the next person to be seated is one who has the highest relationship with the most recently seated person, ignoring any effects of future choices.

The omission of a rule such as "maximising locally" from the group's final algorithm may indicate discrepancies between *how the students created the optimal solution* (*arrangement*) and *the students' report on how they created the solution*. The latter might involve reflecting on the solution found and re-creating the former. These discrepancies seem related to Cai et al.'s (1998) distinction between *procedure* and *algorithm*. To recall, *procedure* refers to the entire process by which a solution is found, while *algorithm* refers to communicating this process in a succinct way that utilises recurring steps and patterns in the process. The students in my study, I claim, took the step from procedure to algorithm, as evidenced by the properties of their solution that they noticed (e.g., "no negative relationships") and the manifestation of these properties in their algorithm. However, there properties of the solution that the group did not explicitly notice, but which likely emerged from rules that the group actually used to create the optimal arrangement. The absence of explicit remarks pertaining to, for example "maximising locally" might suggest that the group used such a rule subconsciously, while they were creating their optimal solution. Cai et al. (1998) argued that the transition from procedure to algorithm requires students to understand these rules at a conscious level. I hypothesise that a possible prerequisite for *understanding a rule at a conscious level* is explicitly noticing a property of the solution that closely corresponds to the rule. For example, the rule of "avoiding negatives" corresponded to the "no negative relationships" property that the group explicitly noticed. In contrast, the rule of "maximising locally" could not be traced to an explicitly-noticed property of the solution. Furthermore, the alignment of these explicitly-noticed properties with the group's predominant contextual considerations of aptness suggests the significance of that to which the aptness of the solution and the final algorithm are considered. That is, the sorts of properties explicitly noticed and ultimately reflected in the final algorithm might be influenced by those aspects of the task that the students deem particularly important to address.

Discrepancies between *how the students created the solution* and *the students' report on how they created the solution* can be construed as a challenge that students might face when they engage with algorithmic tasks. As such, what can be done to help students externalise more faithfully the rules (which at times are used subconsciously) that actually govern how they create their solution? Two lines of suggestions come to mind. First, the question can be approached from a task design perspective (e.g., Watson & Ohtani, 2015) that focuses on developing questions in the task that would help elicit these subconscious rules by, for instance, directing their attention to particular aspects of the solution they have created. Alternatively, the question can be approached from a metacognitive perspective (Schoenfeld, 1985) which focuses on developing students' awareness of the rules they are using, while they are using it to create a solution. Exploring these alternative (complementary) approaches in the context of students engaging with algorithmic tasks warrants further research.

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